c_p

BY

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ABSTRACT

The space c_p is the class of operators on a Hilbert space for which the c_p norm $|T|_p=[\text{trace}(T^* T)^{p/2}]^{1/p}$ is finite. We prove many of the known results concerning c_p in an elementary fashion, together with the result (new for $1 < p < 2$) that c_p is as uniformly convex a Banach space as l_p. In spite of the remarkable parallel of norm inequalities in the spaces c_p and l_p, we show that $p \neq 2$, no c_p built on an infinite dimensional Hilbert space is equivalent to any subspace of any l_p or L_p space.

1. Introduction. This paper is devoted to a systematic study of the classes of compact operators on a Hilbert space known as c_p. Briefly, c_p is the linear space of those operators T for which $|T|_p = [\text{tr}(T^*T)^{p/2}]^{1/p}$ is finite. We show that there is a complete parallel between the spaces of operators c_p and the sequence spaces l_p, all the more surprising because no non-trivial c_p is isometric to any subspace of any l_p or L_p space nor is an infinite dimensional c_p even bicontinuously imbeddable in any l_p or L_p space by a linear map. Our principal new result is that for $1 < p < 2$, c_p is uniformly convex and has the same modulus of convexity as l_p. In spite of the remarkable parallels in norm inequalities in the theory of c_p and l_p spaces — (analogues of the Hölder and Minkowski inequalities as well as Clarkson's inequalities which verify the uniform convexity of l_p) — c_p and l_p are very different as Banach spaces. The situation seems to be that extremal cases of the norm inequalities studied occur in commutative *-subalgebras of c_p which are necessarily isometric to l_p. The non-commutativity of c_p as an algebra (operator multiplication) seems to serve only to make the proofs of the theorems more involved. Most of our other results are not new, but we hope that our techniques are of some interest in themselves. We use principally the spectral theorem for self-adjoint operators and the polar decomposition. In particular we do not prove theorems in the finite dimensional case and then pass to a limit, nor do we make explicit use of the concept of tensor product, nor of the more sophisticated interpolation techniques. Most of the known results concerning c_p may be found in Gohberg and Krein [5], Dunford and Schwartz [4, pp. 1088–1144], Grothendieck [6], Dixmier [2], and Schatten [13, 14], and the references therein. The beginnings of the subject, together with a number of related special theorems,

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seems to be [11]. The spaces we consider are called S_p by Gohberg and Krein, C_p by Dunford and Schwartz, and $\mathscr{L}_p$ by Dixmier. We have chosen c for compact, lower case because the inclusion relations between the spaces c_p are those of l_p, not L_p. In fact, L_p has operator analogues which are spaces consisting mostly of unbounded operators; we shall study these in a later paper. Gohberg and Krein study the Orlicz space analogues of these spaces; the ratio of complexity to novelty remains high.

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Throughout this paper H will denote a fixed Hilbert space with norm $|\cdot|$ and inner product $(\cdot,\cdot)$; the dimension of H is unimportant. p will range in the interval $0 < p \leqq \infty$, and for $1 \leqq p \leqq \infty$, p' will always denote the conjugate exponent to p: $(1/p)+(1/p')=1$. For linear operators A,B on H, we write $A \geqq B$ to mean that A and B are both self-adjoint and $(Ax,x) \geqq (Bx,x)$ for all x in H. We say A is positive if $A \geqq 0$.

We shall make frequent use of the polar decomposition of an operator, but in forms which are not quite usual; thus we include this development. We denote the range of an operator T by $\mathscr{R}(T)$ and the null space of T by $\mathscr{N}(T)$; recall that $\mathscr{R}(T)^\perp = \mathscr{N}(T^*)$, $\mathscr{N}(T)^\perp = \overline{\mathscr{R}(T^*)}$.

First notice that for any x in H, $|Tx|^2 = (T^*Tx,x) = ((T^*T)^{1/2}x, (T^*T)^{1/2}x) = |(T^*T)^{1/2}x|^2$. Thus $\eta(T) = \eta((T^*T)^{1/2}$, and upon taking orthogonal complements, $\overline{\mathscr{R}(T^*)} = \overline{\mathscr{R}((T^*T)^{1/2})}$. Define U to be that linear operator for which $U[(T^*T)^{1/2}y+z] = Ty$ $(y \perp \mathscr{N}(T),\, z \in \mathscr{N}(T))$, elements of the form $(T^*T)^{1/2}y+z$ are dense in H and U has bound 1, so U may be extended uniquely to an operator (again denoted by U) on all of H. We also note that $T = U(T^*T)^{1/2}$ and further, $UT^*T = TT^*U$. It follows immediately by induction that $U(T^*T)^n = (TT^*)^nU$ for all integral $n \geqq 0$; thus for any polynomal ϕ, and hence for any Borel function ϕ, $U\phi(T^*T) = \phi(TT^*)U$. In particular, with $\phi(t) = t^{1/2}$ we have $T = U(T^*T)^{1/2} = (TT^*)^{1/2}U$ and with $\phi(t) = t^{1/4}$ we have $T = U(T^*T)^{1/4}(T^*T)^{1/4} = (TT^*)^{1/4}U(T^*T)^{1/4}$. It may also be shown that $U^*U\phi(T^*T) = \phi(T^*T) = \phi(T^*T)U^*U$ for any Borel function ϕ which vanishes at zero.

We will also use the trace of an operator. Suppose that an operator A is either positive or satisfies $\sum_\alpha |(A\phi_\alpha,\phi_\alpha)| < \infty$ for some orthonormal basis $\{\phi_\alpha\}$ of H. Then if $\{\psi_\beta\}$ is any other orthonormal basis of H, the interchange of the order of summation is permissible in

$$\begin{aligned}\sum_\alpha (A\phi_\alpha,\phi_\alpha) &= \sum_\alpha \sum_\beta (A\psi_\beta,\psi_\beta)\,|(\phi_\alpha,\psi_\beta)|^2 \\ &= \sum_\beta (A\psi_\beta,\psi_\beta) \sum_\alpha |(\phi_\alpha,\psi_\beta)|^2 = \sum_\beta (A\psi_\beta,\psi_\beta).\end{aligned}$$

Thus the quantity $\sum_\alpha (A\phi_\alpha, \phi\ )$ is independent of the orthonormal basis $\{\phi_\alpha\}$ of H; we call this quantity the trace of A, denoted $\operatorname{tr} A$. If A is positive and compact, we may choose $\{\phi_\alpha\}$ to be an orthonormal basis for H consisting of eigenvectors for A; thus $\operatorname{tr} A$ is simply the sum of the eigenvalues of A enumerated with their multiplicities. It is also true that if $\sum_\alpha |(A\phi_\alpha, \phi_\alpha)| < \infty$ for an orthonormal basis $\{\phi_\alpha\}$ of H, that the eigenvalues of such an A are absolutely summable and that their sum is $\operatorname{tr} A$, but we will neither use nor need this fact.

Now suppose that T is a compact operator on H. The operator T^*T is positive and compact and has a unique positive square root which is also compact. The characteristic numbers of T are defined to be the eigenvalues μ_α of $(T^*T)^{1/2}$ enumerated with their multiplicity; we arrange them in a decreasing sequence, at most countably many being greater than zero, as

$$\mu_1(T) \geqq \mu_2(T) \geqq \cdots \geqq 0, \qquad \mu_n(T) \to 0.$$

For $0 < p < \infty$, we define $|T|_p$ then *c_p-norm of T*, whether finite or infinite, to be

$$|T|_p = \left\{ \sum_{n=1}^{\infty} [\mu_n(T)]^p \right\}^{1/p} = \left\{ \sum_\alpha [\mu_\alpha(T)]^p \right\}^{1/p} = [\operatorname{tr}(T^*T)^{p/2}]^{1/p}.$$

We set $|T|_\infty$ to be simply the operator norm of T. The class c_p is the set of all T for which $|T|_p$ is finite.

At this point, we wish to observe three facts which will be used throughout this work. Their proofs are sufficiently immediate to be omitted.

LEMMA 1.1 a. $|T|_p = |(T^*T)^{1/2}|_p$.

b. *If $A \geqq 0$, and r is a positive number, then $|A\ |_{p/r} = |A|_p^r$.*

c. *If $p \leqq q$, $|T|_p \geqq |T|_q$.*

We now prove the preliminary theorem that $|T|_p = |T^*|_p$.

LEMMA 1.2. *Let U be unitary. Then UT and TU have the same characteristic numbers as T. Thus for every p, $|T|_p = |UT|_p = |TU|_p$.*

Proof. The squares of the characteristic numbers of UT are the eigenvalues of $(UT)^*(UT) = T^*U^*UT = T^*T$, so that $\mu_\alpha^2(UT) = \mu_\alpha^2(T)$; since characteristic numbers are non-negative, it follows that $\mu_\alpha(UT) = \mu_\alpha(T)$. The squares of the characteristic numbers of TU are the eigenvalues of $(TU)^*(TU) = U^*TT^*TU$; this operator is unitarily equivalent to, hence has the same eigenvalues as, T^*T.

THEOREM 1.3. *T and T^* have the same characteristic numbers (zero possibly excepted). Thus for every p, $|T|_p = |T^*|_p$.*

Proof. It is a well-known fact that for any two elements a, b in a Banach algebra (such as $a = T$, $b = T^*$ in the algebra of all bounded operators on H), the spectrum of ab is equal to the spectrum of ba (zero possibly excepted). For our purposes however, we need also to keep track of the multiplicity of the spectrom; the theorem may be demonstrated by means of a perturbation argument from this general Banach-algebraic fact, but we prefer to give the following alternative proof.

If H is finite-dimensional, then in the polar decomposition of $T: T = U(T^*T)^{1/2}$ we may take U to be unitary. Thus $T = U(T^*T)^{1/2}$ and $T^* = (T^*T)^{1/2}U^*$; by lemma 1.2, both T and T^* have the same characteristic numbers as $(T^*T)^{1/2}$. If H is not finite dimensional, the operator U appearing in the polar decomposition of T need not be unitary nor admit of replacement by a unitary. Thus we adopt the following procedure which is valid for H of any dimension.

Consider the Hilbert space $\hat{H}$ which is the direct sum of H with itself: $\hat{H} = H \oplus H$. Let $\hat{T} = T \oplus 0$ so that $\hat{T}^*\hat{T} = T^*T \oplus 0$; thus T and $\hat{T}$ have the same characteristic numbers (zero possibly excepted). Similarly T^* and $\hat{T}^*$ have the same characteristic numbers (zero possibly excepted). Let U be the partial isometry which appears in the polar decomposition of $T: T = U(T^*T)^{1/2}$. U is an isometry of $\overline{\mathscr{R}((T^*T)^{1/2})}$ onto $\overline{\mathscr{R}(T)}$ and vanishes on $\mathscr{N}((T^*T)^{1/2})$. The operator $\hat{U} = U \oplus 0$ then maps $\mathscr{R}((\hat{T}^*\hat{T})^{1/2})$ isometrically onto $\overline{\mathscr{R}(\hat{T})}$ and vanishes on $\mathscr{N}((\hat{T}^*\hat{T})^{1/2})$ The orthogonal complements of $\mathscr{R}((\hat{T}^*\hat{T})^{1/2})$: $\mathscr{R}((T^*T)^{1/2})^{\perp} \oplus H$, and of $\mathscr{R}(\hat{T})$: $\mathscr{R}(T)^{\perp} \oplus H$, are of the same dimension and thus there exists an isometry $\hat{V}$ of $\mathscr{R}((\hat{T}^*\hat{T})^{1/2})^{\perp}$ onto $\mathscr{R}(\hat{T})^{\perp}$ which vanishes on $\mathscr{R}((\hat{T}^*\hat{T})^{1/2}$. The operator $\hat{W} = \hat{U} + \hat{V}$ is an isometry of $\hat{H}$ onto $\hat{H}$ and hence is unitary; also $\hat{T} = \hat{W}(\hat{T}^*\hat{T})^{1/2}$, hence $\hat{T}^* = (\hat{T}^*\hat{T})^{1/2}\hat{W}^*$. By Lemma 1.2, the characteristic numbers of $\hat{T}$ are those of $(\hat{T}^*\hat{T})^{1/2}$ which are those of $\hat{T}^*$. Thus (zero possibly excepted), the characteristic numbers of T are those of T^*.

2. Norm inequalities in c_p.

In this section we will prove analogues of the Hölder and Minkowski inequalities for c_p as well as Clarkson's inequalities which demonstrate the uniform convexity of $c_p (1 < p < \infty)$. The importance of our first lemma cannot be over-emphasized. While it is no more than an elementary observation, it is the result which underlies all our computations.

LEMMA 2.1. *Let $A \geqq 0$. Let $x \in H$. Let γ be a given positive real number. Then*

$$\text{if } 0 < \gamma \leqq 1, \quad (A^\gamma x, x) \leqq (Ax, x)^\gamma |x|^{2(1-\gamma)};$$

$$\text{if } 1 \leqq \gamma < \infty, \quad (A^\gamma x, x) \geqq (Ax, x)^\gamma |x|^{2(1-\gamma)}.$$

If $\gamma \neq 1$, equality implies that x is an eigenvector of A.

Proof. First suppose $\gamma \geqq 1$. Let $E(\cdot)$ denote the spectral resolution of A. Then using the Hölder inequality we have

$$
\begin{aligned}
(Ax,x) &= \int_0^\infty \lambda(E(d\lambda)x,x) \\
&\leqq \left[\int_0^\infty \lambda^\gamma(E(d\lambda)x,x)\right]^{1/\gamma} \cdot \left[\int_0^\infty 1\cdot(E(d\lambda)x,x)\right]^{(\gamma-1)/\gamma} \\
&= (A^\gamma x,x)^{1/\gamma}\cdot |x|^{2((\gamma-1)/\gamma)}
\end{aligned}
$$

If $0<\gamma\leqq 1$, then apply what has just been proved to the operator A^γ and the number $1/\gamma:(A^\gamma x,x) \leqq (A^\gamma/\gamma)x,x)^\gamma |x|^{2(1-\gamma)}$. Equality in the case $\gamma\neq 1$ requires that the ratio of λ^γ to 1 be constant on the support of the measure $(E(d\lambda)x,x)$; this support may thus contain at most one point λ_0 so we have $Ax=\lambda_0 x$.

As an immediate consequence of Lemma 2.1 we obtain the following useful expressions for the c_p norm of an operator.

LEMMA 2.2.* *If* $0<p\leqq 2$, *then* $|T|_p^p = \inf \sum_\alpha |T\phi_\alpha|^p$;

if $2\leqq p<\infty$, *then* $|T|_p^p = \sup \sum_\alpha |T\phi_\alpha|^p$.

(The inf or sup is to be taken over all orthonormal bases of H). If $p\neq 2$, equality occurs if and only if $\{\phi_\alpha\}$ is an orthonormal basis for H consisting of eigenvectors for T^*T.

If $p\geqq 1$, then $|T|_p = \sup\{\sum_\alpha |(T\phi_\alpha,\psi_\alpha)|^p\}^{1/p}$ where ϕ_α and ψ_α run over all pairs of orthonormal bases for H. Equality holds if and only if $\{\phi_\alpha\}$ is an is an orthonormal basis for H consisting of eigenvectors for T^*T and $\{\psi_\alpha\}$ an orthonormal basis for H obtained by completing the orthonormal set $\{(T\phi_\alpha/|T\phi_\alpha|): T\phi_\alpha\neq 0\}$.

Proof. Using Lemma 2.1 with $\gamma=p/2$, we have for any orthonormal basis $\{\phi_\alpha\}$ of H that

$$((T^*T)^{p/2}\phi_\alpha,\phi_\alpha) \leqq (T^*T\phi_\alpha,\phi_\alpha)^{p/2} = |T\phi_\alpha|^p \quad (p\leqq 2),$$

$$((T^*T)^{p/2}\phi_\alpha,\phi_\alpha) \leqq (T^*T\phi_\alpha,\phi_\alpha)^{p/2} = |T\phi_\alpha|^p \quad (p\geqq 2).$$

Summing on α, we see that

$$|T|_p^p = \operatorname{tr}(T^*T)^{p/2} \leqq \sum_\alpha |T\phi_\alpha|^p \quad (p\leqq 2),$$

$$|T|_p^p = \operatorname{tr}(T^*T)^{p/2} \geqq \sum_\alpha |T\phi_\alpha|^p \quad (p\geqq 2).$$

* Part of Lemma 2.2 appears in Dunford and Schwartz [3] as Lemma XI.9.32. The condition "$2\leqq p$" given there should read "$p\leqq 2$." The lemma appears *in toto* in Gohberg and Krein [5, p. 155].

The conditions for equality in Lemma 2.1 show us that if $p/2 \neq 1$, $|T|_p^p = \sum_\alpha |T\phi_\alpha|^p$ if and only if every ϕ_α is an eigenvector of T^*T.

The last assertion is proved by using the polar decomposition

$$T = (TT^*)^{1/4}\, U(T^*T)^{1/4},$$

so that

$$\begin{aligned}
\sum_\alpha |(T\phi_\alpha, \psi_\alpha)|^p &= \sum_\alpha |((TT^*)^{1/4}U(T^*T)^{1/4}\phi_\alpha, \psi_\alpha)|^p \\
&\leqq \sum_\alpha |(T^*T)^{1/4}\phi_\alpha|^p\, |(TT^*)^{1/4}\psi_\alpha|^p \\
&\leqq \left\{\sum_\alpha |(T^*T)^{1/4}\phi_\alpha|^{2p}\right\}^{1/2} \left\{\sum_\alpha |(TT^*)^{1/4}\psi_\alpha|^{2p}\right\}^{1/2} \\
&= \left\{\sum_\alpha ((T^*T)^{1/2}\phi_\alpha, \phi_\alpha)^p\right\}^{1/2} \left\{\sum_\alpha ((TT^*)^{1/2}\psi_\alpha, \psi_\alpha)^p\right\}^{1/2} \\
&\leqq \left\{\sum ((T^*T)^{p/2}\phi_\alpha, \phi_\alpha)\right\}^{1/2} \left\{\sum ((TT^*)^{p/2}\psi_\alpha, \psi_\alpha)\right\}^{1/2} \\
&= |T|_p^{p/2}\, |T^*|_p^{p/2} = |T|_p^p.
\end{aligned}$$

The conditions necesary and sufficient for equality may be seen directly from this chain of inequalities.

Unfortunately, this last assertion fails for $p < 1$, as may be seen by considering

$$T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \phi_1 = \psi_1 = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \quad \phi_2 = \psi_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

We may now prove analogues of the Hölder and Minkowski inequalities.

THEOREM 2.3*. *Let* $T \in c_p$, $S \in c_q$. *Then* $TS \in c_r$ *with* $1/r = 1/p + 1/q$, $0 < p, q, r \leqq \infty$, *and* $|TS|_r \leqq |T|_p |S|_q$. *Equality holds if and only if: if* $p, q < \infty$, $(T^*T)^p$ *is a multiple of* $(SS^*)^q$; *if* $p = \infty > q$, $PT^*TP = |T|_\infty^2 P$ *where* P *is the orthogonal projection of* H *onto the range of* SS^*; *if* $p < \infty = q$, $QSS^*Q = |S|_\infty^2 Q$ *where* Q *is the orthogonal projection of* H *onto the range of* T^*T.

Proof. First suppose $p = \infty$, $q = r < \infty$. If $q = r \leqq 2$, let $\{\phi_\alpha\}$ be an orthonormal basis for H consisting of eigenvectors for S^*S. Then we have by Lemma 2.2

$$(2.3,1) \qquad |TS|_q^q \leqq \sum_\alpha |TS\phi_\alpha|^q \leqq |T|_\infty^q \sum_\alpha |S\phi_\alpha|^q = |T|_\infty^q |S|_q^q.$$

To investigate the case of equality, we first notice that for the second inequality to be an equality we must have $|TS\phi_\alpha| = |T|_\infty |S\phi_\alpha|$ for every eigenvector of S^*S. Consider those ϕ_α for which $S^*S\phi_\alpha = \lambda_\alpha \neq 0$ and let the polar decomposition

* Theorem 2.3 appears in A. Horn [7], derived from some inequalities of Weyl [4, p. 1079].

of S be $S = U(S^*S)^{1/2}$. Since U is an isometry on this set of ϕ_α, we have that $\{U\phi_\alpha : \lambda_\alpha \neq 0\} = \{\lambda_\alpha^{-1/2} S\phi_\alpha : \lambda_\alpha \neq 0\}$ is an orthonormal set. Let us denote this set by $\{\psi_\alpha\}$, and notice that $\{\psi_\alpha\}$ spans precisely $\overline{\mathscr{R}(S)} = \overline{\mathscr{R}(SS^*)} = \mathscr{R}(P)$. The equality $|TS\phi_\alpha| = |T|_\infty |S\phi_\alpha|$ yields $|T\psi_\alpha| = |T|_\infty |\psi_\alpha|$; since T attains its norm on each ψ_α, it must attain its norm everywhere on the span of the ψ_α, so that $T/|T|_\infty$ is an isometry on the range of P; this is equivalent to $PT^*TP = |T|_\infty^2 P$. Conversely, the condition $PT^*TP = |T|_\infty^2 P$ implies that $S^*T^*TS\phi_\alpha = S^*PT^*TPS\phi_\alpha = |T|_\infty^2 S^*PS\phi_\alpha = |T|_\infty^2 S^*S\phi_\alpha$ so that if ϕ_α is an eigenvector for S^*S and the second inequality in (2.3.1) is an equality, then necessarily ϕ_α is an eigenvector of T^*S^*ST and the first inequality in (2.3.1) must also be an equality.

Continuing with the case $p = \infty$ $q = r < \infty$, suppose $q \geqq 2$. Let $\{\phi_\alpha\}$ be an orthonormal basis for H consisting of eigenvectors for T^*S^*ST. Then by Lemma 2.2,

$$|TS|_q^q = \sum_\alpha |TS\phi_\alpha|^q \leqq |T|_\infty^q \sum_\alpha |S\phi_\alpha|^q \leqq |T|_\infty^q |S|_q^q. \tag{2.3.2}$$

To have equality throughout forces the last inequality to be an equality. By Lemma 2.2, each ϕ_α must be an eigenvector of S^*S. That the middle inequality of (2.3.2) be an equality is equivalent to $PT^*TP = |T|_\infty^2 P$ just as previously. Conversely, if ϕ_α is an eigenvector of S^*T^*TS and $PT^*TP = |T^*|_\infty^2 P$, then $S^*T^*TS\phi_\alpha = S^*PT^*TPS\phi_\alpha = |T|_\infty^2 S^*S\phi_\alpha$ so that ϕ_α must be an eigenvector of S^*S and the last inequality of 2.3.2 must be an equality.

Next we consider the case $q = \infty$, $p = r < \infty$. The isometry of the adjoint mapping in every c_p class (Theorem 1.3) shows that $|TS|_p^p = |S^*T^*|_p^p \leqq |S^*|_\infty^p |T^*|_p^p = |S|_\infty^p |T|_p^p$, the middle inequality being wht we have just proved. We have also just proved that equality holds if and only if $QSS^*Q = Q(S^*)^*S^*Q = |S|_\infty^2 Q$ where Q is the orthogonal projection of H on $\overline{\mathscr{R}(T^*)} = \overline{\mathscr{R}(T^*T)}$. This completes the proof in the case $p = \infty$ or $q = \infty$, so that throughout the remainder of the proof we take $p < \infty$, $q < \infty$, $r < p, q$.

We next prove the theorem in the case $r \leqq 2$. Together with $r \leqq 2$, let us also assume for the moment that $p \geqq 2$. Let $\{\phi_\alpha\}$ be an orthonormal basis for H consisting of eigenvectors for S^*S: $S^*S\phi_\alpha = \lambda_\alpha^2 \phi_\alpha$. Using the polar decomposition of $S : S = U(S^*S)^{1/2}$ where U is an isometry on the range of S^*S, we see that $\{U\phi_\alpha : \lambda_\alpha \neq 0\}$ is again an orthonormal set; we complete this set of vectors to an orthonormal basis for H, denoted $\{\psi_\alpha\}$. Now we use Lemma 2.2 first for $r \leqq 2$, then for $p \geqq 2$, and the Hölder inequality for sequences $(r/q + r/p = 1)$ to obtain

$$\begin{aligned} |TS|_r^r &\leqq \sum_\alpha |TS\phi_\alpha|^r = \sum_\alpha |TU\lambda_\alpha\phi_\alpha|^r \\ &= \sum_\alpha \lambda_\alpha^r |T\psi_\alpha|^r \leqq \Big(\sum_\alpha \lambda_\alpha^q\Big)^{r/q} \Big(\sum_\alpha |T\psi_\alpha|^p\Big)^{r/p} \\ &= |S|_q^r |T|_p^r. \end{aligned} \tag{2.3.3}$$

If we have equality throughout, in particular we must have equality in the application of the Hölder inequality, which requires that the ratio of $|S\phi_\alpha|^q = \lambda_\alpha^q$ to $|T\psi_\alpha|^p$ be some constant independent of α, generically denoted throughout by c. For the last inequality to be an equality requires ψ_α to be eigenvectors of T^*T. Now $\lambda_\alpha SS^*\psi_\alpha = SS^*S\phi_\alpha = S\lambda_\alpha^2\phi_\alpha = \lambda_\alpha^3\psi_\alpha$ so that ψ_α are also eigenvectors for SS^*. The facts that T^*T and SS^* are positive, have a common basis of eigenvector and that $c|(T^*T)^p\psi_\alpha|^{1/2} = c|T\psi_\alpha|^p = |S^*\psi_\alpha|^q$, together imply $(T^*T)^p = c(SS^*)^q$. Conversely, $(T^*T)^p = c(SS^*)^q$ may be seen to imply equality throughout (2.3.3); we leave the details to the reader.

Continuing with the case $r \leqq 2$, now assume $p \leqq 2$. Using the polar decomposition of T, we have

$$T = U(T^*T)^{1/2n}\cdots(T^*T)^{1/2n},$$

where there are n factors of $(T^*T)^{1/2n}$; n is chosen so that $n \geqq 2/p$. Since $T \in c_p$, $(T^*T)^{1/2n} \in c_{np}$ and $|(T^*T)^{1/2n}|_{np} = |T|_p^{1/n}$(Lemma 1.1). Using what we have already proved ($np \geqq 2$), we have

$$(2.3.4) \quad |(T^*T)^{(k+1)/2n}S|_{1/q+(k+1/pn)^{-1}} \leqq |T_p^{1/n}|\ |(T^*T)^{k/2n}S|_{(1/q+k/n)^{-1}} \qquad (k = 0,1,2,\cdots,n-1),$$

which yields

$$(2.3.5) \qquad |(T^*T)^{1/2}S|_r \leqq |T|_p|S|_q,$$

Since $|U|_\infty \leqq 1$, we have

$$|TS|_r \leqq |U(T^*T)^{1/2}S|r_r \leqq |U|_\infty|T|_p|S|_q = |T|_p|S|_q.$$

Equality holds if and only if equality holds at every step of (2.3.4) and at (2.3.5). Equality at (2.3.4) implies in particular ($k=0$) that $((T^*T)^{1/n})^{np} = c(SS^*)^q$, which may be seen conversely to imply $|TS|_r = |T|_p|S|_q$.

Finally we consider the case $2 < r$. If n is an integer $n \geqq r/2$, then

$$(S^*T^*TS)^n \in c_{r/n}\left(\frac{r}{n} \leqq 2\right)$$

and from what we have already proved, the operators $(S^*T^*TS)^nS^*T^*TS$, $(S^*T^*TS)^nS^*T^*T$, $(S^*T^*TS)^nS^*T^*$ and $(S^*T^*TS)^nS^*$ also belong to various c classes, $t \leqq 2$. After a certain amount of arithmetic, we find

$$|(S^*T^*TS)^{n+1}|_{r/(n+1)} \leqq |(S^*T^*TS)^n|_{r/n}|S^*|_q|T^*|_p|T|_p|S|_q.$$

Using Lemma 1.1 and Theorem 1.3, we have

$$|TS|_r^{2(n+1)} \leqq |TS|_r^{2n}|T|_p^2|S|_q^2$$

from which $|TS|_r \leqq |T|_p|S|_q$. We can have equality only if

$$\{[T^*TS(S^*T^*TS)^n][(S^*T^*TS)^nS^*T^*T]\}^{((2(n+1)/r)-(1/q))^{-1}} = c(SS^*)_q;$$

this (with c generically denoting some constant) is equivalent to

$$S^*\{T^*TS(S^*T^*TS)^{2n}S^*T^*T\}S = S^*(SS^*)^{-1+(2q(n+1)/r)}S = c(S^*S)^{(2q/r)(n+1)},$$

or

$$(S^*T^*TS) = c(S^*S)^{q/r}.$$

Thus the restriction of T^*T to the range of S must be $(SS^*)^{q/r-1} = (SS^*)^{q/p}$; but then T^*T must be zero on the orthogonal complement of $\mathscr{R}(S^*S)$ if we are to have $|TS|_r = |T|_p|S|_q$, so we must in fact have $(T^*T)^p = c(SS^*)^q$. That the condition $(T^*T)^p = c(SS^*)^q$ is sufficient for $|TS|_r = |T|_p|S|_q$ may again be easily verified and the theorem is complete.

THEOREM 2.4. *Let $T,S \in c_p$, $1 \leqq p < \infty$. Then $|T+S|_p \leqq |T|_p + |S|_p$, so that c_p is a linear space and $|\cdot|_p$ is a norm on c_p. If $p > 1$, equality holds if and only if $aT = bS$ for some $a,b \geqq 0$, $a+b>0$. If $p=1$, equality holds if and only if T^*T and S^*S have a common set of eigenvectors $\{\phi_\alpha\}$ which form an orthonormal basis for H such that for each α there exist $a_\alpha, b_\alpha \geqq 0$, $a_\alpha + b_\alpha > 0$, with $a_\alpha T\phi_\alpha = b_\alpha S\phi_\alpha$,*

Proof. From Lemma 2.2, let $\{\phi_\alpha\}$, $\{\psi_\alpha\}$ be two orthonormal bases for H such that

$$|T+S|_p = \left\{\sum_\alpha |((T+S)\phi_\alpha,\psi_\alpha)|^p\right\}^{1/p}.$$

The Minkowski inequality for sequences yields

$$|T+S|_p \leqq \left\{\sum_\alpha |(T\phi_\alpha,\psi_\alpha)|^p\right\}^{1/p} + \left\{\sum_\alpha |(S\phi_\alpha,\psi_\alpha)|^p\right\}^{1/p},$$

and by Lemma 2.2 again, this is dominated by $|T|_p + |S|_p$. If we have $|T+S|_p = |T|_p + |S|_p$, we see first from Lemma 2.2 that ϕ_α must be eigenvectors of both T^*T and S^*S, and that (unless $T\phi_\alpha = 0$)$\psi_\alpha = T\phi_\alpha/|T\phi_\alpha|$ and (unless $S\phi_\alpha = 0$) $\psi_\alpha = S\phi_\alpha/|S\phi_\alpha|$. Equality in the use of the Minkowski inequality forces, for $p>1$, constants a, $b \geqq 0$ $a+b>0$, such that $a(T\phi_\alpha,\psi_\alpha) = b(S\phi_\alpha,\psi_\alpha)$; since $(T\phi_\alpha,\psi_\beta) = 0 = (S\phi_\alpha,\psi_\beta)$ for $\alpha \neq \beta$, we must have $aT = bS$. Conversely, it is clear that these conditions imply $|T+S|_p = |T|_p + |S|_p$. For $p=1$, equality in the use of the Minkowski inequality forces only constants $a_\alpha, b_\alpha \geqq 0$, $a_\alpha + b_\alpha > 0$, such that $a_\alpha(T\phi_\alpha,\psi_\alpha) = b_\alpha(S\phi_\alpha,\psi_\alpha)$; again $(T\phi_\alpha,\psi_\beta) = 0 = (S\phi_\alpha,\psi_p)$ for $\alpha \neq \beta$ shows $a_\alpha T\phi_\alpha = b_\alpha S\phi_\alpha$ for all α. Conversely these conditions are surely sufficient to imply $|T_1 + S|_1 = |T|_1 + |S|_1$.

We now continue with the estimates which show that $c_p(1 < p < \infty)$ is uniformly convex. Recall that a normed linear space $X = \{x\}$ is said to be uniformly convex if the function

$$\delta(\varepsilon) = \inf_{\substack{|x|=|y|=1, \\ |x-y|=\varepsilon}} \left(1 - \frac{1}{2}|x+y|\right)$$

is strictly positive in some range $0 < \varepsilon < \varepsilon_0$. J. A. Clarkson [1] showed that the sequence spaces $l_p(1 < p < \infty)$ are uniformly convex by proving a number of sharp inequalities concerning the norm of elements in l_p; those which imply that l_p is strictly convex are:

$$|x+y|^{p'} + |x-y|^{p'} \leqq (|x|^p + |y|^p)^{p/p'} \quad (1 < p \leqq 2),$$

and

$$|x+y|^p + |x-y|^p \leqq 2^{p-1}(|x|^p + |y|^p) \quad (2 \leqq p < \infty).$$

If we take $|x| = |y| = 1$, we have

$$|x+y|^{p'} \leqq 2^{p'} - |x-y|^{p'} \quad (1 < p \leqq 2),$$

$$|x+y|^p \leqq 2^p - |x-y|^p \quad (2 \leqq p < \infty).$$

The modulus of convexity is thus given by

$$2\delta(\varepsilon) = 2 - (2^{p'} - \varepsilon^{p'})^{1/p'} \quad (1 < p \leqq 2),$$

$$2\delta(\varepsilon) = 2 - (2^p - \varepsilon^p)^{1/p} \quad (2 \leqq p < \infty).$$

We will now show that c_p has the same modulus of convexity as l_p by demonstrating that Clarkson's inequalities hold in c_p. The conditions under which equality holds will simply be stated without proof; they only require checking the cases of equality in the inequalities within the proofs.

Dixmier [2] used an interpolation theorem to show that c_p and l_p have the same modulus of convexity in the range $2 \leqq p < \infty$; our technique in this range uses only the Hölder inequality. In view of the fact that the only known proof Clarkson's inequality for $l_p(1 < p < 2)$ seems to be the lengthly original demonstration of Clarkson, it is not suprising that the analogous result for c_p must be at least as troublesome.

We first state for reference some inequalities concerning real numbers. The proofs of the first two may be safely left to the reader.

Inequality a: Let $-a \leqq b < a$. Then

if $0 < \gamma \leqq 1$, $2^{\gamma-1}(a^\gamma + b^\gamma) \leqq (a+b)^\gamma + (a-b)^\gamma \leqq 2(a^\gamma + b^\gamma)$;

if $1 \leqq \gamma < \infty$, $2(a^\gamma + b^\gamma) \leqq (a+b) + (a-b)^\gamma \leqq 2^{\gamma-1}(a^\gamma + b^\gamma)$.

Inequality b: Let $a \geqq 0$ $b \geqq 0$. Then

$$\text{if } 0 < \gamma \leqq 1, \quad (a+b)^\gamma \leqq a^\gamma + b^\gamma;$$

$$\text{if } 1 \leqq \gamma < \infty, \quad (a+b)^\gamma \geqq a^\gamma + b^\gamma.$$

The third inequality which we require is the very deep inequality of Clarkson [1]. The forms in which we use this are

Inequality c: Let $-a_\alpha \leqq b_\alpha \leqq a$. Then for $1 < p \leqq 2$

$$\left\{\sum_\alpha (a_\alpha + b_\alpha)^p\right\}^{p'/p} + \left\{\sum_\alpha (a_\alpha - b_\alpha)^p\right\}^{p'/p} \leqq 2\left\{\sum_\alpha a_\alpha^p + |b_\alpha|^p\right\}^{p'/p}$$

Let $-a \leqq b \leqq a$. Then for $2 \leqq p < \infty$

$$|a+b|^{p'} + |a-b|^{p'} \geqq 2(a^p + |b|^p)^{p'/p}$$

We now generalize these numerical inequalities to inequalities for c_p norms of operators.

LEMMA 2.5. *Suppose A, B are operators on H and $-A \leqq B \leqq A$. Then*

$$\text{if } 0 < \gamma \leqq 1, \quad \operatorname{tr}(A+B)^\gamma + \operatorname{tr}(A-B)^\gamma \leqq 2 \operatorname{tr} A^\gamma;$$

$$\text{if } 1 \leqq \gamma < \infty, \quad \operatorname{tr}(A+B)^\gamma + \operatorname{tr}(A-B)^\gamma \geqq 2 \operatorname{tr} A^\gamma.$$

If $\gamma \neq 1$ and the quantities involved are finite, then equality holds if and only if $AB = 0$.

Proof. The lemma has non-trivial content only if A and B are compact; also A and B must be self-adjoint and A positive. Let $\{\phi_\alpha\}$ be an orthonormal basis for H consisting of eigenvectors for $A: A\phi_\alpha = \lambda_\alpha \phi_\alpha$. Since $-(A\phi_\alpha, \phi_\alpha) \leqq (B\phi_\alpha, \phi_\alpha) \leqq (A\phi_\alpha, \phi_\alpha)$, inequality a yields

$$((A+B)\phi_\alpha, \phi_\alpha)^\gamma + ((A-B)\phi_\alpha \phi_\alpha)^\gamma \leqq 2(A\phi_\alpha, \phi_\alpha)^\gamma = 2\lambda_\alpha^\gamma \qquad (\gamma \leqq 1),$$

$$((A+B)\phi_\alpha, \phi_\alpha)^\gamma + ((A-B)\phi_\alpha, \phi_\alpha)^\gamma \geqq 2(A\phi_\alpha, \phi_\alpha)^\gamma = 2\lambda_\alpha^\gamma \qquad (\gamma \geqq 1).$$

Also, $A + B \geqq 0$ and $A - B \geqq 0$, so Lemma 2.1 yields

$$((A+B)^\gamma \phi_\alpha, \phi_\alpha) + ((A-B)^\gamma \phi_\alpha, \phi_\alpha)$$

$$\leqq ((A+B)\phi_\alpha, \phi_\alpha)^\gamma + ((A-B)\phi_\alpha, \phi_\alpha)^\gamma \leqq 2\lambda_\alpha^\gamma \qquad (\gamma \leqq 1)$$

$$((A+B)^\gamma \phi_\alpha, \phi_\alpha) + ((A-B)^\gamma \phi_\alpha, \phi_\alpha)$$

$$\geqq ((A+B)\phi_\alpha, \phi_\alpha)^\gamma + ((A-B)\phi_\alpha, \phi_\alpha)^\gamma \geqq 2\lambda_\alpha^\gamma \qquad (\gamma \geqq 1).$$

Summing on α we finally obtain

$$\operatorname{tr}(A+B)^{\gamma}+\operatorname{tr}(A-B)^{\gamma} \leqq 2\sum_{\alpha} \lambda^{\gamma} = 2\operatorname{tr} A^{\gamma} \qquad (\gamma \leqq 1),$$

$$\operatorname{tr}(A+B)^{\gamma}+\operatorname{tr}(A-B)^{\gamma} \geqq 2\sum_{\alpha} \lambda^{\gamma} = 2\operatorname{tr} A^{\gamma} \qquad (\gamma \geqq 1).$$

LEMMA 2.6. *Suppose A, B are operators on H and $A \geqq 0$, $B \geqq 0$. Then*

$$\text{if } 0<\gamma \leqq 1, \quad \operatorname{tr}(A+B)^{\gamma} \leqq \operatorname{tr} A^{\gamma}+\operatorname{tr} B^{\gamma};$$

$$\text{if } 1 \leqq \gamma<\infty, \quad \operatorname{tr}(A+B)^{\gamma} \geqq \operatorname{tr} A^{\gamma}+\operatorname{tr} B^{\gamma}.$$

If $\gamma \neq 1$ and the quantities involved are finite, equality holds if and only if $AB=0$.

Proof. First we find operators C, D on H such that $C(A+B)^{1/2}=A^{1/2}$, $D(A+B)^{1/2}=B^{1/2}$ and $C^*C+D^*D=I$. [We do this as follows: $B \geqq 0$ implies $A \leqq A+B$, so $|A^{1/2}x|=(Ax,x)^{1/2} \leqq ((A+B)x,x)^{1/2}=|(A+B)^{1/2}x|$; similarly $|B^{1/2}x| \leqq |(A+B)^{1/2}x|$. Let $x=(A+B)^{1/2}u+v$, where $v \in \mathcal{N}(A+B)$, $u \perp \mathcal{N}(A+B)$; such x are dense in H, so C and D are uniquely determined if we require $Cx=A^{1/2}u$, $Dx=B^{1/2}u+v$. Clearly $C(A+B)^{1/2}=A^{1/2}$ and $D(A+B)^{1/2}=B^{1/2}$; $C^*C+D^*D=I$ follows from $|Cx|^2+|Dx|^2=|A^{1/2}u|^2+|B^{1/2}u|^2+|v|^2=(Au,u)+(Bu,u)+|v|^2=|(A+B)^{1/2}u|^2+|v|^2=|x|^2$. Note also that $|Cx| \leqq |x|$, $|Dx| \leqq |x|$]. Then we have $(A+B)^{\gamma}=(A+B)^{\gamma/2}C^*C(A+B)^{\gamma/2}+(A+B)^{\gamma/2}D^*D(A+B)^{\gamma/2}$, and

$$\begin{aligned}
\operatorname{tr}(A+B)^{\gamma} &= |C(A+B)^{\gamma/2}|_2^2+|D(A+B)^{\gamma/2}|_2^2 \\
&= |(A+B)^{\gamma/2}C^*|_2^2+|(A+B)^{\gamma/2}D^*|_2^2 \\
&= \operatorname{tr} C(A+B)^{\gamma}C^*+\operatorname{tr} D(A+B)^{\gamma}D^*.
\end{aligned}$$

To estimate $\operatorname{tr} C(A+B)^{\gamma}C^*$, let $\{\phi_{\alpha}\}$ be an orthonormal basis for H consisting of eigenvectors for $A: A\phi_{\alpha}=\lambda_{\alpha}\phi_{\alpha}$. Then by Lemma 2.1,

$$\begin{aligned}
&(C(A+B)^{\gamma}C^*\phi_{\alpha},\phi_{\alpha}) \\
&\quad \leqq ((A+B)C^*\phi_{\alpha},C^*\phi_{\alpha})^{\gamma}|C^*\phi_{\alpha}|^{2(1-\gamma)} \leqq (A\phi_{\alpha},\phi_{\alpha})^{\gamma}=\lambda_{\alpha}^{\gamma} \quad (\gamma \leqq 1), \\
&(C(A+B)^{\gamma}C^*\phi_{\alpha},\phi_{\alpha}) \\
&\quad \geqq ((A+B)C^*\phi_{\alpha},C^*\phi_{\alpha})^{\gamma}|C^*\phi_{\alpha}|^{2(1-\gamma)} \geqq (A\phi_{\alpha},\phi_{\alpha})^{\gamma}=\lambda_{\alpha}^{\gamma} \quad (\gamma \geqq 1).
\end{aligned}$$

(Recall that $|C^*\phi_{\alpha}|^{2(1-\gamma)} \leqq |\phi_{\alpha}|^{2(1-\gamma)}=1$ for $\gamma \leqq 1$, and $|C^*\phi_{\alpha}|^{-2(1-\ )} \leqq |\phi_{\alpha}|^{-2\ 1-\gamma)}=1$ for $\gamma \geqq 1$). Thus summing on α yields

$$\operatorname{tr} C(A+B)^{\gamma}C^* \leqq \operatorname{tr} A^{\gamma} \qquad (\gamma \leqq 1),$$

$$\operatorname{tr} C(A+B)^{\gamma}C^* \geqq \operatorname{tr} A^{\gamma} \qquad (\gamma \geqq 1).$$

Similarly,

$$\operatorname{tr} D(A+B)^{\gamma} D^* \leqq \operatorname{tr} B^{\gamma} \qquad (\gamma \leqq 1),$$

$$\operatorname{tr} D(A+B)^{\gamma} D^* \geqq \operatorname{tr} B^{\gamma} \qquad (\gamma \geqq 1).$$

and the lemma is proved.

Lemmas 2.5 and 2.6 yield all of Clarkson's inequalities for $c_p(1 < p < \infty)$, which we formulate as:

THEOREM 2.7. *Let T, S be operators on H. Then*

(i) . $2^{p-1}(|T|_p^p+|S|_p^p) \leqq |T+S|_p^p + |T-S|_p^p \leqq 2(|T|_p^p+|S|_p^p) \qquad (0 < p \leqq 2),$

(ii). $|T+S|_p^{p'}+|T-S|_p^{p'} \leqq 2(|T|_p^p+|S|_p^p)^{p'/p} \qquad (1 < p \leqq 2),$

(iii). $2(|T|_p^p+|S|_p^p) \leqq |T+S|_p^p+|T-S|_p^p \leqq 2^{p-1}(|T|_p^p+|S|_p^p) \qquad (2 \leqq p < \infty),$

(iv). $2(|T|_p^p+|S|_p^p)^{p'/p} \leqq |T+S|_p^{p'}+|T-S|_p^{p'} \qquad (2 \leqq p < \infty).$

If $p = 2$, equality always holds; if $p \neq 2$ and the quantities involved are finite, equality holds in (i) or (iii) if and only if $T^*TS^*S = 0$, in (ii) or (iv) if and only if $T = S$ or $T = 0$ or $S = 0$.

Proof of (*i*) *and* (*iii*): First consider $p \leqq 2$. Then $|T+S|_p^p+|T-S|_p^p = \operatorname{tr}((T^*T+S^*S)+(T^*S+S^*T))^{p/2} + \operatorname{tr}((T^*T+S^*S) - (T^*S+S^*T))^{p/2}$. Lemma 2.5 is applicable with $A = T^*T+S^*S$, $B = T^*S+S^*T$, $\gamma = p/2 \leqq 1$, for $|(Bx,x)| = 2|\operatorname{Re}(Sx,Tx)| \leqq 2|Sx|\ |Tx| \leqq |Sx|^2+|Tx|^2 = (Ax,x)$; thus $|T+S|_p^p+|T-S|_p^p \leqq \operatorname{tr}(T^*T+S^*S)^{p/2}$. Now apply Lemma 2.6 with $A = T^*T$, $B = S^*S$, $\gamma = p/2 \leqq 1$ to obtain

$$|T+S|_p^p+|T-S|_p^p \leqq 2[\operatorname{tr}(T^*T)^{p/2} + \operatorname{tr}(S^*S)^{p/2}] = 2(|T|_p^p+|S|_p^p).$$

It follows from this that also

$$|2T|_p^p+|2S|_p^p \leqq 2(|T+S|_p^p+|T-S|_p^p)$$

so that

$$2^{p-1}(|T|_p^p+|S|_p^p) \leqq |T+S|_p^p+|T-S|_p^p \leqq 2(|T|_p^p+|S|_p^p) \quad (0 < p \leqq 2).$$

(iii) follows in exactly the same way, with the sense of all inequalities reversed.

Proof of (*ii*). Let A and B be as above, and let $\{\phi_\alpha\}$ be an orthonormal basis for H. We use inequality c with a_α, b_α defined $a_\alpha \pm b_\alpha = ((A \pm B)^{p/2}\phi_\alpha, \phi_\alpha)^{1/p}$, to obtain

$$\begin{aligned} &|T+S|_p^{p'}+|T-S|_p^{p'} \\ &\quad\leqq 2\Big\{\sum_\alpha 2^{-p}\Big[((A+B)^{p/2}\phi_\alpha,\phi_\alpha)^{1/p} + ((A-B)^{p/2}\phi_\alpha,\phi_\alpha)^{1/p}\Big]^p \\ &\qquad + 2^{-p}\big|((A+B)^{p/2}\phi_\alpha,\phi_\alpha)^{1/p} - ((A-B)^{p/2}\phi_\alpha,\phi_\alpha)^{1/p}\big|^p\Big\}^{p'/p}. \end{aligned}$$

Apply inequality a to each of the summands on the right-hand side (with $\gamma = p > 1$) to obtain

$$
\begin{aligned}
&|T+S|_p^{p'} + |T-S|_p^{p} \\
&\leqq 2 \sum_\alpha 2^{-p} \cdot 2^{p-1}[((A+B)^{p/2}\phi_\alpha, \phi_\alpha) + ((A-B)^{p/2}\phi_\alpha, \phi_\alpha)]\}^{p'/p} \\
&\leqq 2\{2^{-1}[\operatorname{tr}(A+B)^{p/2} + \operatorname{tr}(A-B)^{p/2}]\}^{p'/p}.
\end{aligned}
$$

Use of Lemma 2.6, then Lemma 2.5 (with $\gamma = p/2 \leqq 1$) yields

$$
\begin{aligned}
|T+S|_p^{p'} + |T-S|_{p'}^{p'} &\leqq 2(\operatorname{tr} A^{p/2})^{p'/p} = 2\{\operatorname{tr}(T^*T + S^*S)^{p/2}\}^{p'/p} \\
&\leqq 2\{\operatorname{tr}(T^*T)^{p/2} + \operatorname{tr}(S^*S)^{p/2}\}^{p'/p} = 2(|T|_p^p + |S|^p)^{p'/p}
\end{aligned}
$$

Proof of (iv). Again set $A = T^*T + S^*S$, $B = T^*S + S^*T$, and let $\{\phi_\alpha\}$ be an orthonormal basis for H. Since $p \geqq 2$, $p/p' \geqq 1$, and the triangle inequality in $l_{p'/p}$ yields

$$
\begin{aligned}
&|T+S|_p^{p'} + |T-S|_p^{p'} \\
&= \left\{\sum ((A+B)^{p/2}\phi_\alpha, \phi_\alpha)\right\}^{p'/p} + \left\{\sum ((A-B)^{p/2}\phi_\alpha, \phi_\alpha)\right\}^{p'/p} \\
&\geqq \sum_\alpha [((A+B)^{p/2}\phi_\alpha, \phi_\alpha)^{p'/p} + ((A-B)^{p/2}\phi_\alpha, \phi_\alpha)^{p'/p}]^{p/p'}\}^{p'/p}.
\end{aligned}
$$

Inequality a, with $\gamma = p'$, yields

$$
\begin{aligned}
&((A+B)^{p/2}\phi_\alpha, \phi_\alpha)^{p'/p} + ((A-B)^{p/2}\phi_\alpha, \phi_\alpha))^{p'/p} \\
&\quad 2\left\{\left[\frac{((A+B)^{p/2}\phi_\alpha, \phi_\alpha)^{1/p} + ((A-B)^{p/2}\phi_\alpha, \phi_\alpha)^{1/p}}{2}\right]^{p'} \right. \\
&\qquad \left. + \left|\frac{((A+B)^{p/2}\phi_\alpha, \phi_\alpha)^{1/p} - ((A-B)^{p/2}\phi_\alpha, \phi_\alpha)^{1/p}}{2}\right|^{p'}\right\}
\end{aligned}
$$

Application of inequality c with $p \geqq 2$ yields

$$
\begin{aligned}
&((A+B)^{p/2}\phi_\alpha, \phi_\alpha)^{p'/p} + ((A-B)^{p/2}\phi_\alpha, \phi_\alpha)^{p'/p} \\
(2.7.2) \qquad &\geqq 2 \cdot 2 \left(\left[\frac{((A+B)^{p/2}\phi_\alpha, \phi_\alpha)^{1/p}}{2}\right]^p + \left[\frac{((A-B)^{p/2}\phi_\alpha, \phi_\alpha)^{1/p}}{2}\right]^p\right)^{p'/p} \\
&= 2 \cdot 2^{1-p'}[((A+B)^{p/2}\phi_\alpha, \phi_\alpha) + ((A-B)^{p/2}\phi_\alpha, \phi_\alpha)]^{p'/p}.
\end{aligned}
$$

The insertion of (2.7.2) into (2.7.1) then yields

$$
\begin{aligned}
&|T+S|_p^{p'} + |T-S|_p^{p'} \geqq 2 \cdot 2^{1-p'}\{\operatorname{tr}(A+B)^{p/2} + \operatorname{tr}(A-B)^{p/2}\}^{p'/p} \\
&\geqq 2 \cdot 2^{1-p'}\{2 \operatorname{tr} A^{p/2}\}^{p'/p} = 2(\operatorname{tr} A^{p/2})^{p'/p} \geqq 2(|T|_p^p + |S|_p^p)^{p'/p}
\end{aligned}
$$

the last two inequalities being applications of Lemmas 2.6 and 2.5 with $\gamma = p/2 \geqq 1$.

To conclude this section, we prove the following special result for $0 < p < 1$. Professor Gohberg has informed us that this is a special case of a theorem of S. Ju. Rotfel'd [to appear in *Functional Analysis and its Applications*, I, fasc. 3 (1967)].

THEOREM 2.8. $|T + S|_p^p \leqq |T|_p^p + |S|_p^p$, $(0 < p \leqq 1)$. *If* $p < 1$ *and* T *and* S *are in* c_p, *equality holds if and only if* $T^*TS^*S = 0$. *Thus, if* $p \leqq 1$, $\rho(S, T) = |T - S|_p^p$ is a metric on c_p.

Proof. First we demonstrate the theorem in the case $0 < p \leqq 2/3$. Since $(T^*T)^{1-p}$ and $(S^*)^{1-p}$ are both dominated by $(T^*T)^{1-p} + (S^*S)^{1-p}$, there are operators C and D, both of operator norm at most one, such that

$$(T^*T)^{(1-p)/2} = C[(T^*T)^{1-p} + (S^*S)^{1-p}]^{1/2}$$

and

$$(S^*S)^{(1-p)/2} = D[(T^*T)^{1-p} + (S^*S)^{1-p}]^{1/2}.$$

(cf. the proof of Lemma 2.6).

Using the polar decompositions of the operators T and S we have

$$\begin{aligned} T + S &= U(T^*T)^{1/2} + V(S^*S)^{1/2} \\ &= [U(T^*T)^{p/2}C + V(S^*S)^{p/2}D][(T^*T)^{1-p} + (S^*S)^{1-p}]^{1/2}. \end{aligned}$$

Using our generalized Hölder inequality (Theorem 2.3) followed by the triangle inequality in c_1 (Theorem 2.4), we have

$$\begin{aligned} |T + S|_p &\leqq |U(T^*T)^{p/2}C + V(S^*S)^{p/2}D|_1 |[(T^*T)^{1-p} + (S^*S)^{1-p}]^{1/2}|_{p(1-p)} \\ &\leqq \{|U(T^*T)^{p/2}C|_1 + |V(S^*S)^{p/2}D|_1\} \\ &\qquad \{\mathrm{tr}[(T^*T)^{1-p} + (S^*S)^{1-p}]^{p/2(1-p)}\}^{(1-p)/p}. \end{aligned}$$

Using the fact that the operator norms of C, D, U and V are at most one, we have

$$\begin{aligned} |T + S|_p &\leqq \{|(T^*T)^{p/2}|_1 + |(S^*S)^{p/2}|_1\}\{\mathrm{tr}[(C^*T)^{1-p} + (S^*S)^{1-p}]^{p/2(1-p)}\}^{(1-p)/p} \\ &= (|T|_p^p + |S|_p^p)\{\mathrm{tr}[(T^*T)^{1-p} + (S^*S)^{1-p}]^{p/2(1-p)}\}^{(1-p)/p}. \end{aligned}$$

Since $0 < p \leqq 2/3$, $p/2(1-p) \leqq 1$, so that by Lemma 2.6

$$\mathrm{tr}[(T^*T)^{1-p} + (S^*S)^{1-p}]^{p/2(1-p)} \leqq \mathrm{tr}(T^*T)^{p/2} + \mathrm{tr}(S^*S)^{p/2} = |T|_p^p + |S|_p^p.$$

We thus have

$$|T + S|_p^p \leqq (|T|_p^p + |S|_p^p)(|T|_p^p + |S|_p^p)^{(1-p)/p} = (|T|_p^p + |S|_p^p)^{1/p}.$$

Now consider the case $2/3 < p < 1$. Let Q denote the operator $[(T^*T)^{p/2}$

$+ (S^*S)^{p/2}]^{1/2}$ and let C, D be operators such that $C^*C+D^*D=I$, $(T^*T)^{p/4} = CQ = QC^*(S^*S)^{p/4} = DQ = QD^*$. The restriction $2/3 < p < 1$ allows us to write

$$\begin{aligned} T+S &= U(T^*T)^{1/2} + V(S^*S)^{1/2} \\ &= [U(CQ)^{2/p-2}QC^*CQ^{3-2/p} + V(DQ)^{2/p-2}\, QD^*DQ^{3-2/p}]Q^{2/p-2}. \end{aligned}$$

Using our generalized Hölder inequality, the triangle inequality in c_1, and the facts that $|U|_\infty \leqq 1$, $|V|_\infty \leqq 1$, we have

$$\begin{aligned} &|T+S|_p \\ &\leqq |U(CQ)^{2/p-2}QC^*CQ^{3-2/p} + V(DQ)^{2/p-2}QD^*DQ^{3-2/p}]Q^{2/p-2}|_1 |Q^{2/p-2}|_{p/1-p} \\ &\leqq \{|(CQ)^{2/p-2}QC^*CQ^{3-2/p}|_1 + |(DQ)^{2/p-2}\, QD^*DQ^{3-2/p}|_1\}\, |Q^2|_1^{1-p/p} \end{aligned}$$

Assuming $|(CQ)^{2/p-2}QC^*CQ^{3-2/p}|_1 \leqq |QC^*CQ|_1$ and $|(CQ)^{2/p-2}QD^*DQ^{3-2/p}|_1 \leqq |QD^*DQ|_1$, which we will prove in a moment, we have

$$\begin{aligned} |T+S|_p &\leqq \{\operatorname{tr} QC^*CQ + \operatorname{tr} QD^*DQ\}\, |Q^2|_1^{1-p/p} \\ &= (\operatorname{tr} Q(C^*C+D^*D)Q \cdot (\operatorname{tr} Q^2)^{1-p/p} \\ &= (\operatorname{tr} Q^2)^{1/p} = (|T|_p^p + |S|_p^p)^{1/p}. \end{aligned}$$

To show that $|(CQ)^{2/p-2}QC^*CQ^{3-2/p}|_1 \leqq |QC^*CQ|_1$, we note first that

$$\begin{aligned} |(CQ)^{2/p-2}QC^*CQ^{3-2/p}|_1 &\leqq |(CQ)^{2(1-p/p)}|_{p/1-p}\, |QC^*CQ^{3-2/p}|_{p/2p-1} \\ &= |QC^*CQ|_1^{1-p/p} \cdot |Q^{3-2/p}\, C^*CQ|_{p/2p-1}. \end{aligned}$$

Let now $\{\phi_\alpha\}$ be an orthonormal basis for H consisting of eigenvectors for QC^*CQ: $QC^*CQ\phi_\alpha = \lambda_\alpha\phi_\alpha$. For $2/3 < p < 1$, $1 < p/2p-1 < 2$ and $0 < 3-2/p < 1$, so that Lemmas 2.2 and 2.1 yield

$$\begin{aligned} |Q^{3-2/p}C^*CQ|_{p/2p-1}^{p/2p-1} &\leqq \sum_\alpha (Q^{2(3-2/p)}C^*CQ\phi_\alpha, C^*CQ\phi_\alpha)^{p/2(2p-1)} \\ &\leqq \sum_\alpha (Q^2C^*CQ\phi_\alpha, C^*CQ\phi_\alpha)^{3p-2/2(2p-1)}|C^*CQ\phi_\alpha|^{2-2p/2p-1}. \end{aligned}$$

Since $|C^*|_\infty \leqq 1$, and $2-2p/2p-1>0$, $|C^*CQ\phi_\alpha|^{2-2p/2p-1} \leqq |CQ\phi_\alpha|^{2-2p/2p-1} = \lambda_\alpha^{1-p/2p-1}$. Also, $(Q^2C^*CQ\phi_\alpha, C^*CQ\phi_\alpha)^{3p-2/2(2p-1)} = |QC^*CQ\phi_\alpha|^{3p-2/2p-1} = \lambda_\alpha^{3p-2/2p-1}$. It follows that

$$|Q^{3-2/p}C^*CQ|_{p/2p-1}^{p/2p-1} \sum_\alpha \sum \lambda_\alpha^{3p-2/2p-1}\, \lambda_\alpha^{1-p/2p-1} = \sum_\alpha \lambda_\alpha,$$

so that

$$\begin{aligned} |(CQ)^{2/p-2}\, QC^*CQ^{3-2/p}|_1 &\leqq \Big(\sum_\alpha \lambda_\alpha\Big)^{1-p/p} \Big(\sum \lambda_\alpha\Big)^{2p-1/p} \\ &= \sum_\alpha \lambda_\alpha = |QC^*CQ|_1. \end{aligned}$$

This demonstration used only the fact $|C|_\infty \leqq 1$, so that C may be everywhere replaced by D.

3. **c_p is complete.** Having shown that c_p is a metric space we proceed to prove that c_p is complete in this metric. First notice that since $|T|_p \geqq |T|_\infty$, any Cauchy sequence $\{T_n\}$ in the metric of c_p must be a Cauchy sequence in operator norm. It follows that T_n must converge uniformly to an operator T. Lemma 3.1 below will show then that $T \in c_p$ and that $|T - T_n|_p \to 0$.

LEMMA 3.1. *Let T_n be a sequence of operators which converges uniformly to T. Then $|T|_p \leqq \liminf_{n\to\infty} |T_n|_p$.*

Proof. Since T_n converges uniformly to T, T_n^* converges uniformly to T^* and $A_n = T_n^* T_n$ thus converges uniformly to $A = T^*T$. It follows that for any fixed $\gamma > 0$ A_n^γ converges uniformly to A^γ. [To see this, let $M = \sup |A_n|_\infty$, and let $p(t)$ be a polynomial such that $|p(t) - t^\gamma| \leqq \varepsilon$ for $0 \leqq t \leqq M$. Then $|p(A_n) - A_n^\gamma|_\infty \leqq \varepsilon$ and $|p(A) - A^\gamma|_\infty \leqq \varepsilon$, and since A_n converges uniformly to A, $p(A_n)$ converges uniformly to $p(A)$; thus $|A_n^\gamma - A^\gamma|_\infty \leqq 3\varepsilon$ for all n sufficiently large]. Now let $\{\phi_\alpha\}$ be any orthormal basis for H and let σ be any finite set of indices. Since A_n^γ converges to A^γ, $|A^\gamma\phi_\alpha|^2$ converges to $|A^\gamma\phi_\alpha|^2$ and thus

$$\begin{aligned}\sum_{\alpha\in\sigma} |A^\gamma\phi_\alpha|^2 &= \lim_{n\to\infty} \sum_{\alpha\in\sigma} |A_n^\gamma\phi_\alpha|^2 \\ &\leqq \liminf_{n\to\infty} \sum_{\alpha} |A_n\phi_\alpha|^2 = \liminf |A_n^\gamma|_2^2\end{aligned}$$

Since this holds for all finite sets σ, we must have

$$|A^\gamma|_2^2 = \sup_\sigma \sum_{\alpha\in\sigma} |A^\gamma\phi_\alpha|^2 \leqq \liminf_{n\to\infty} |A_n^\gamma|_2^2 .$$

If we take $\gamma = p/4$, we then have

$$|T|_p^p = |A^{p/4}|_2^2 \leqq \liminf_{n\to\infty} |A_n^{p/4}|_2^2 = \liminf_{n\to\infty} |T_n|_p^p.$$

COROLLARY 3.2. *c_p is complete.*

Proof. Let $\{T_n\}$ be a Cauchy sequence in c_p and let T be the operator to which T_n converges uniformly. Then for each fixed n, the operators $T_n - T_m$ converge uniformly to $T_n - T$ as $m \to \infty$. Using Lemma 3.1, we have $|T_n - T|_p \leqq \liminf_{m\to\infty} |T_n - T_m|_p$. Since the sequence $\{T_n\}$ is Cauchy in c_p, $\lim_{n\to\infty} \liminf_{m\to\infty} |T_n - T_m|_p = 0$ and thus $|T_n - T|_p \to 0$.

4. $c_p^* = c_{p'}$. With the completeness of c_p proved, we see that c_p is a Banach space for $p \geqq 1$. Also, we have demonstrated the uniform convexity of c_p for $1 < p < \infty$ and thus, by a Theorem of Pettis [12], c_p must be reflexive for $1 < p < \infty$. The norm inequalities of Theorem 2.7 show that the norm in c_2 satisfies the parallelogram law and thus c_2 is a Hilbert space. We now show that in a natural

way the dual space to c_p is $c_{p'}$ $(1 < p < \infty)$. We first show that operators in c_1 possess a trace.

LEMMA 4.1. *If $T \in c_1$ then* $\operatorname{tr} T$ *exists and* $|\operatorname{tr} T| \leqq |T|_1$.

Proof. Lemma 2.2 actually shows that if $\{\phi\}$, $(\psi\}$ are any two orthonormal bases for H, then $\sum_\alpha (T\phi_\alpha, \psi_\alpha)| \leqq |T|_1$. In particular, we may take $\phi_\alpha = \psi_\alpha$.

Our next two theorems show that $c_p^* = c_{p'}$ in a natural way. First we show $c_{p'} \subset c_p^*$ and then $c_p^* \subset c_{p'}$.

THEOREM 4.2. *Let $1 < p < \infty$, and let $S \in c_{p'}$. Then $F(T) = \operatorname{tr}(ST)$ is a continuous linear functional on c_p with norm precisely $|S|_{p'}$, attained when $T = V^*(SS^*)^{p'/p}$ (V is the partial isometry occuring in the polar decomposition of S).*

Proof. For any $T \in c_p$, ST is in c_1 (Lemma 2.3) and $|ST|_1 \leqq |S|_{p'}|T|_p$ so that $\operatorname{tr}(ST)$ exists and may be computed by $\operatorname{tr} ST = \sum_\alpha (ST\phi_\alpha, \phi_\alpha)$ for any orthonormal basis $\{\phi_\alpha\}$ of H. The fact that $\operatorname{tr} ST$ may be so computed shows that $\operatorname{tr} S(T_1 + T_2) = \operatorname{tr} ST_1 + \operatorname{tr} ST_2$ and $\operatorname{tr} S(aT) = a \operatorname{tr} ST$ so that $\operatorname{tr} ST$ is in fact linear on c_p. That $\operatorname{tr} ST$ is bounded with bound $|S|_{p'}$ follows from $|\operatorname{tr} ST| \leqq |S|_{p'}|T|_p$. It is easy to check that when $T = V^*(SS^*)^{p'/p}$, $\operatorname{tr} ST = |S|_{p'}|T|_p$.

THEOREM 4.3. *Let $F(T)$ be any bounded linear functional on $c_p(1 < p < \infty)$. Then there exists an S in $c_{p'}$ such that $F(T) = \operatorname{tr} ST$. (By Theorem 4.2, $|S|_{p'}$ is the bound of F; it is also clear that S is uniquely defined by F).*

Proof. It is no loss of generality to assume that F has bound 1. Let $T_n \in c_p$ be chosen so that $|T_n|_p = 1$ and $F(T_n) \to 1$. Then $F(T_n + T_m) \leqq |T_n + T_m|_p$, but as $n, m \to \infty$, $F(T_n + T_m) \to 2$. Since c_p is uniformly convex, $|T_n - T_m|_p \to 0$ and thus T_n converges to some operator $T \in c_p$ with $|T|_p = 1$. We need now only to show that $F(R) = \operatorname{tr} SR$ for every $R \in c_p$. Consider the functional G on c_p given by $G(R) = F(R) - \operatorname{tr} SR$. Let $a = 1/2|G|$ and let A be in c_p, $|A|_p = 1$ such that $G(A) = 2a$. [That G attains its bound follows from the uniform convexity of c_p ust as did the fact that F attains its bound]. We then have $F(A) = z + a$, $\operatorname{tr} SA = z - a$ for some complex number z. Consider first the case $1 < p < 2$. Since $|F| = 1$, $|S|_{p'} = 1$, $|A|_p = 1$ and $|T|_p = 1$, we have using Theorem 2.7 (i),

$$
\begin{aligned}
&|F(T + \varepsilon A)|^p + |\operatorname{tr} S(T - \varepsilon A)|^p \\
(4.3.1)\qquad &\leqq |T + \varepsilon A|_p^p + |T - \varepsilon A|_p^p \leqq 2(1 + \varepsilon^p)(\varepsilon > 0, 1 < p \leqq 2).
\end{aligned}
$$

The left-hand side of (4.3.1) is

$$
\begin{aligned}
\{(1 + \varepsilon \operatorname{Re} z + \varepsilon a)^2 + (\varepsilon \operatorname{Im} z)^2\}^{p/2} &+ \{(1 - \varepsilon \operatorname{Re} z + \varepsilon a)^2 + (\varepsilon \operatorname{Im} z)^2\}^{p/2} \\
&= 2 + 2p\varepsilon a + 0(\varepsilon^2) \quad \text{as } \varepsilon \to 0.
\end{aligned}
$$

Thus $2 + 2p\varepsilon a + 0(\varepsilon^2) \leqq 2(1 + |\varepsilon|^p)$. Letting ε tend to zero and recalling that $p > 1$, > we see that $a = 0$. That is to say, $|G| = 0$ and thus $F(R)$ is in fact equal to $\operatorname{tr} SR$ for all $R \in c_p$.

Now consider the case $p > 2$. We have again from Theorem 2.7 (iv) that

$$
\begin{aligned}
&|F(T + \varepsilon A)|^{p'} + |\operatorname{tr}(S(T - \varepsilon A)|^{p'} \\
(4.3.2) \qquad &\leqq |T + \varepsilon A|_p^{p'} + |T - \varepsilon A|_p^{p'} \leqq \left\{\frac{1}{2}[|2T|_p^{p'} + |2\varepsilon A|_p^{p'}]\right\}^{p/p'} \\
&= 2(1 + \varepsilon^{p'})^{p'/p}.
\end{aligned}
$$

The left-hand side of (4.3.2) is equal to

$$
\begin{aligned}
&\{(1 + \varepsilon \operatorname{Re} z + \varepsilon a)^2 + (\varepsilon \operatorname{Im} z)^2\}^{p'/2} + \{(1 - \varepsilon \operatorname{Re} z + \varepsilon a)^2 + (\varepsilon \operatorname{Im} z)^2\}^{p'/2} \\
&\qquad = 2 + 2p'\varepsilon a + 0(\varepsilon^2) \qquad (\varepsilon \to 0),
\end{aligned}
$$

while the right-hand side of (4.3.2) is equal to $2 + 2^{p/p'}\varepsilon^{p'} + 0(\varepsilon^{2p'})$. Again as $\varepsilon \to 0$, recalling that $p < \infty$ so that $p' > 1$, we see that $a = |G| = 0$ and thus that $F(R) = \operatorname{tr} SR$ for all R.

5. Concluding remarks. We have exhibited a remarkable parallel between the spaces of operators c_p and the sequence spaces l_p. In fact, all our norm inequalities for c_p contain the same inequalities for l_p; to see this, it suffices to show that there exists an isometric imbedding of l_p into c_p: Let $\{\phi_\alpha\}$ be any orthonormal basis for H and let P_α be the orthogonal projection of H onto the span of ϕ_α defined by $P_\alpha \phi_\alpha = \delta_{\alpha\beta}\phi_\beta$. For every sequence $\xi = \{\xi_\alpha\} \in l$, let $T = T_\xi = \sum_\alpha \xi_\alpha P_\alpha$ Then $T^*T = \sum_\alpha |\xi_\alpha|^2 P_\alpha$, so the characteristic numbers of T are $|\xi_\alpha|$ and $|T|_p^p = \sum_\alpha |\xi_\alpha|^p = |\xi|_{l_p}^p$.

Since c_p shares ao many properties in common with l_p, one would like to be reassured that c_p is not, in fact, an l_p or L_p space or even some subspace thereof. Of course c_2 is a Hilbert space, but for $p \neq 2$, Lemma 1.2 shows us that there are far too many isometries of c_p for c_p to be an l_p or L_p space. Perhaps the most convincing demonstration is to actually calculate some norms when $p \neq 2$.

Any class c_p of operators on a Hilbert space of dimension exceeding 1 has a subspace which may be represented as 2×2 matrices of the form $\begin{pmatrix} ab \\ cd \end{pmatrix}$. Suppose that this subspace of c_p were isometric with some four-dimensional subspace of some space $L_p(\Omega, d\mu)$, with $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ corresponding to the function $af + bg + ch + dk$. By taking $a = 1$, $b = c = d = 0$ and equating the c_p norm of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and the L_p norm of f, we have $\int_\Omega |f|^p = 1$; similarly $\int_\Omega |g|^p = \int_\Omega |h|^p = \int_\Omega |k|^p = 1$.

Next we consider $\begin{pmatrix}1 & 0\\ 0 & e^{i\theta}\end{pmatrix}$ to see that $2 = \int_\Omega |f + e^{i\theta}k^p|$ for every θ. Integrating in θ and using the Hölder inequality, we have for $p < 2$

$$\begin{aligned} 2 &= \int_\Omega \int_0^{2\pi} \frac{d\theta}{2\pi} |f + e^{i\theta}k|^p \leqq \int_\Omega \left(\int_0^{2\pi} \frac{d\theta}{2\pi} |f + e^{i\theta}k|^2 \right)^{p/2} \\ &= \int_\Omega (|f|^2 + |k|^2)^{p/2} \leqq \int_\Omega (|f|^p + |k|^p) = 2. \end{aligned}$$

In particular, to have equality in the last inequality above for $p < 2$, f and k must have disjoint supports. Similarly, the supports of g and h must be disjoint. If $p > 2$, the sense of all inequalities above is reverrsed but the conclusion is the same. Now let Ω_1 be the intersection of the supports of f and g, Ω_2 that of f and h, Ω_3 that of g and k, and Ω_4 that of h and k; thus $\Omega_1, \cdots, \Omega_4$ are mutually disjoint. By taking $c = d = 0$ and equating norms, we see that

$$(|a|^2 + |b|^2)^{1/2} = \int_{\Omega_1} |af + bg|^p + \int_{\Omega_2} |a|^p |f|^p + \int_{\Omega_3} |b|^p |g|^p.$$

Thus $\int_{\Omega_1} |af + bg|^p$ is a function only of $|a|$ and $|b|$; similarly, $\int_{\Omega_2} |af + ch|_p$ is a function only of $|a|$ and $|c|$, $\int_{\Omega_3} |bg + dk|^p$ is a function only of $|b|$ and $|d|$, and $\int_\Omega |ch+dk|^p$ is a function only of $|c|$ and $|d|$. Thus $\int_\Omega |af+bg+ch+dk|^p = \int_{\Omega_1} |af + bg|^p + \cdots + \int_{\Omega_4} |ch + dk|^p$ is a function only of $|a|$, $|b|$, $|c|$, and $|d|$. The presumed isometric imbedding of c_p into L_p thus yields the equality of the c_p norms of

$$\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \text{ and } \begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix},$$

or $4 = 2^{p/2} + 2^{p/2}$, which is impossible for $p \neq 2$.

We have seen that for $p \neq 2$, c_p on a two-dimensional Hilbert space is not isometric with any subspace of any L_p space. Of course between any two Banach spaces of the same finite dimension there exists a bicontinuous linear transformation. We now show that if $p \neq 2$, there is no bicontinous map between c_p on an infinite dimensional Hilbert space and any subspace of any L_p space. In passing, we will obtain estimates on how far from an isometry any linear one-to-one map between c_p on a finite-dimensional Hilbert space and any subspace of any L_p must be. Our example is derived from that of S. Kakutani [8].

Let $\{\phi_\alpha\}$ be an orthonormal basis for H, fixed once and for all, and denote by P_α the orthogonal projection on H defined by $P_\alpha \phi_\beta = \delta_{\alpha\beta}\phi_\alpha$; the operator norm of $\sum_\alpha a_\alpha P_\alpha$ is $\sup_\alpha |a_\alpha|$. Define the operators E_α and F_α on c_p by $E_\alpha(T) = P_\alpha T$, $F_\alpha(T) = TF_\alpha$; by Theorem 2.3, $|\sum_\alpha a_\alpha E_\alpha(T)|_p \leqq |\sum_\alpha a_\alpha P_\alpha|_\infty |T|_p = \sup_\alpha |a_\alpha| |T|_p$ and similarily for the $\{F_\alpha\}$. Thus $\{E_\alpha\}$ and $\{F_\alpha\}$ are the atoms respectively of two Boole an algebras $\{E\}$, $\{F\}$ of projections of bound one on c_p. Let $\{G\}$ denote the Boolean algebra of projections on c_p generated by $\{E\}$ and $\{F\}$. We first obtain estimates for the norms of some elements of $\{G\}$. If we think of operators T in c_p as given by matrices $(t_{\alpha\beta}) = ((T\phi_\alpha, \phi_\beta))$, we see that an operator on c_p of the form $\sum_{\alpha,\beta} a_{\alpha,\beta} E_\alpha F_\beta$ carries $(t_{\alpha\beta})$ into $(a_{\alpha,\beta} t_{\alpha\beta})$. Suppose that T is given by $t_{\alpha\beta} = 1$ $(1(1 \leqq \alpha, \beta \leqq n)$, 0 other wise; and U is given by $u_{\alpha\beta} = n^{-1/2}\omega^{\alpha+\beta}(1 \leqq \alpha, \beta \leqq n)$, 0 otherwise, where ω is a primitive n-th root of unity. T is simply $n^{1/2}Q$ with Q a self-adjoint projection of rank one, and hence $|T|_p = n^{1/2}$; U is the direct sum of an $n \times n$ unitary with zero and hence $|U|_p = n^{1/p}$. It follows that for $0 < p < 2$ the operator $\sum_{\alpha\beta=1,1}^{n,n} \omega^{\alpha+\beta} E_\alpha F_\beta$ on c_p has norm at least $n^{(1/p-1/2)}$.

Now suppose that A is a linear one-to-one operator from c_p into some subspace of an L_p space. Then $\{AEA^{-1}\}$ and $\{AFA^{-1}\}$ are Boolean algebras of projections on some subspace of an L_p space of bound at most $\| A \| \| A^{-1} \|$. It follows from [10] in the case $2 < p < \infty$, or better with the estimates of [9, Section 6] valid uniformly in the range $0 < p < 2$, that $\{AGA^{-1}\}$ is a Boolean algebra of projections with bound at most const. $\| A \|^2 \| A^{-1} \|^2$, and hence that $\{G\}$ has bound at most const. $\| A \|^3 \| A^{-1} \|^3$. But we have already shown that $\{G\}$ has bound at least $n^{(1/p-1/2)}$, for any n no greater that the dimension of H, and thus we must have $\| A \| \| A^{-1} \| \geqq \text{const.} (\dim H)^{1/3(1/p-1/2)}$. In the case that H is infinite-dimensional, we see that A cannot be bicontinuous; in the case that H is finite-dimensional, we have a lower bound for $\| A \| \| A^{-1} \|$. (The constant in this last estimate may be taken to be 14^{-5} uniformly for $0 < p < 2$, although much better constants are undoubtedly available; we also except that the exponent $1/3(1/p - 1/2)$ may be improved to $(1/p - 1/2)$ but no more.) The analogous result for $2 < p \leqq \infty$ follows from the consideration of adjoints, yielding $\| A \| \| A^{-1} \| \geqq \text{const.}$ $(\dim H)^{1/3(1/2-1/p)}$.

There is a partial converse to the Hölder inequality for sequences which states: If $\{a_n\}$ is a sequence such that $\{a_n b_n\} \in l_r$ for every $b_n \in l_p$, then

$$\{a_n\} \in l_q \cdot \left(\frac{1}{r} = \frac{1}{p} + \frac{1}{q}\right).$$

Although everywhere defined, linear, but not continuous, functionals abound on l_p, this theorem says that there are no such functionals which are given by sequences. A similar statement holds for c_p:

THEOREM 5.1. *Let T be an everywhere defined linear operator on H such that $TS \in c_r$ for every $S \in c_p$. Then $T \in c_q$ $(1/r = 1/p + 1/q)$.*

Proof. We first show that T is bounded. If T is not bounded, then there exists an orthonormal set $\{\phi_n\}$ of H such that $|T\phi_n| > 3^n$. [To see this, it is clear that we can choose ϕ_1. Having selected $\phi_1, \cdots, \phi_{n-1}$ orthonormal, select ϕ_n of norm 1 in the orthogonal complement of the subspace of H spanned by $\phi_1, \cdots, \phi_{n-1}$ such that $|T\phi_n| \geqq 3^n \max_{1 \leqq \nu < n} |T\phi_\nu|$; if this cannot be done, the linearity of T alone shows that T must be bounded]. Now define S to be the continuous linear operator for which $S\phi_n = 2^{-n}\phi_n$, $S = 0$ on the orthogonal complement of

$\{\phi_n\}$. S belongs to every c_p $(p > 0)$. But $TS \in c_r$ says in particular that $\infty > |TS\phi_n| \geqq (3/2)^n$ uniformly in n which is impossible. Thus T must be bounded. If $r = q$, this is clearly also all that can be said. Now suppose $r < q$. We assert that T must be compact, for if T were not compact, then T^*T would not be compact and thus by decomposing the spectrum of T^*T (with its multiplicity) we could find a countable orthonormal set $\{\phi_n\}$ in H for which the support of the vectors ϕ_n are disjoint and bounded away from zero; thus, $(T\phi_n, T\,\phi_m) = 0$ if $n \neq m$, $\inf_n |T\phi_n| = a > 0$. Complete $\{\phi\ \}$ to an orthonormal basis $\{\phi_\alpha\}$ for H in any manner whatever. Define S by $S\phi_n = b_n\phi_n$ for the originally chosen ϕ_n's, $S\phi_\alpha = 0$ otherwise, where $\{b_n\} \in l_p$. Then $|TS\phi_n| = |b_n||T\phi\ | \geqq a|b_n|$. If we take, however, $\{b_n\}$ to be in l_q but not in l_r, we see that $\sum |TS\phi_n{}^r| = \infty$. Since $(S^*T^*TS\phi_\alpha, \phi_\beta) = 0$ unless $\alpha = \beta$, we see that $\{\phi_\alpha\}$ must be an orthonormal basis of eigenvectors for S^*T^*TS and hence, by Lemma 2.2, TS cannot be in c_r, contrary to our hypothesis. Finally, knowing that T is compact, let $\{\phi_\alpha\}$ be an orthonormal basis for H consisting of eigenvectors for T^*T. Define S by $S\phi_\alpha = b_\alpha\bar{\phi}_\alpha$ where $\{b_\alpha\} \in l_p$. Then $\{\phi_\alpha\}$ is also an orthonormal basis of eigenvectors for S^*T^*TS and we have $\infty > |TS|_r^r = \sum |TS\phi_\alpha|^r = \sum |b_\alpha|^r |\,T\,\phi_\alpha|^r$. By the theorem on sequences, $\{|T\phi_\alpha|\} \in l_q$ and Lemma 2.2 again yields $T \in c_q$.

The conclude, we prove the often used lemma that operators of finite rank are dense in every c_p.

LEMMA 5.2. *Let $T \in c_p$. Then for every $\varepsilon > 0$, there exists an operator T_ε such that the range of T_ε is finite dimensional and $|T_\varepsilon - T|_p < \varepsilon$.*

Proof. Let μ_α be the characteristic numbers of T and let $\{\phi_\alpha\}$ be an orthonormal basis for H consisting of eigenvectors for T^*T, with $T^*T\phi_\alpha = \mu_\alpha^2\phi_\alpha$. Let ε be chosen arbitrarily small and let σ be a finite set of indices such that $\sum_{\alpha\notin\sigma}\mu_\alpha^p < \varepsilon^p$. Define T_ε by $T_\varepsilon\phi_\alpha = T\phi_\alpha$ $(\alpha \in \sigma)$, $T_\varepsilon\phi_\alpha = 0$ $(\alpha \notin \sigma)$. Then $|T_\varepsilon - T|_p^p = \sum_\alpha |(T_\varepsilon - T)\phi_\alpha|^p = \sum_{\alpha\notin\sigma}\mu_\alpha^p < \varepsilon^p$.

REFERENCES

1. J. A. Clarkson. *Uniformly convex spaces*, Trans. Amer. Math. Soc. **40** (1936), 396–414.

2. J. Dixmier, *Formes linéaires sur un anneau d'operateurs*, Bull. Soc. Math. France **81** (1953) 9–39.

3. N. Dunford, *Spectral Operators*, Pacific J. Math. **4** (1954), 321–354.

4. N. Dunford and J. T. Schwartz. *Linear Operators, Part II.* Interscience, New York (1963).

5. I. C. Gohberg and M. G. Krein. *Introduction to the Theory of Linear Non-Self-Adjoint Operators in Hilbert Space*, Moscow (1965). (In Russian).

6. A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs Amer. Math. Soc. **16** (1955).

7. A. Horn, *On the singular values of a product of completely continuous operators*, Proc. Nat. Acad. Sci. U.S.A. **36** (1950), 373–375.

8. S. Kakutani, *An example concerning uniform boundedness of spectral measures*, Pacific J. Math. **4** (1954), 363–372.

9. W. Littman, C. McCarthy, and N. Rivière, *L^p Multiplier theorems of Marcinkiewicz type*, To appear.

10. C. McCarthy, *Commuting Boolean algebras of projections II*, Proc. Amer. Math. Soc. **15** (1964), 781–787.

11. J. von Neumann. *Some matrix-inequalities and metrization of matric-space*, Tomsk Univ. Rev. **1** (1937), 286–300.

12. B. J. Pettis. *A proof that every uniformly convex space is reflexive*, Duke Math. J. **5** (1939), 249–253

13. R. Schatten, *A theory of cross-spaces*, Ann. of Math. Studies, No. 26, Princeton University Press, Princeton, 1950.

14. ———, *Norm ideals of completely continuous operators*, Ergebnisse der Math, Neue Folge, **27**, Springer Verlag, 1960.

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